Introduction to Representation

Grades PreK–2

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The Math Process Standards Series
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In order to be effective mathematicians, students need to develop understanding of critical math content. They need to understand number and operations, algebra, measurement, geometry, and data analysis and probability. Through continued study of these content domains, students gain a comprehensive understanding of mathematics as a subject with varied and interconnected concepts. As math teachers, we attempt to provide students with exposure to, exploration in, and reflection about the many skills and concepts that make up the study of mathematics.

Even with a deep understanding of math content, however, students may lack important skills that can assist them in their development as effective mathematicians. Along with content knowledge, students need an understanding of the processes used by mathematicians. They must learn to problem solve, communicate their ideas, reason through math situations, prove their conjectures, make connections between and among math concepts, and represent their mathematical thinking. Development of content alone does not provide students with the means to explore, express, or apply that content. As we strive to develop effective mathematicians, we are challenged to develop both students’ content understanding and process skills.

The National Council of Teachers of Mathematics (2000) has outlined critical content and process standards in its *Principles and Standards for School Mathematics* document. These standards have become the roadmap for the development of textbooks, curriculum materials, and student assessments. These standards have provided a framework for thinking about what needs to be taught in math classrooms and how various skills and concepts can be blended together to create a seamless math curriculum. The first five standards outline content standards and expectations related to number and operations, algebra, geometry, measurement, and data analysis and probability. The second five standards outline the process goals of problem solving, reasoning and proof, communication, connections, and representations. A strong understanding of these standards empowers teachers to identify and select activities within their curricula to produce powerful learning. The standards provide a vision for what teachers hope their students will achieve.
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Foreword

This book is a part of a vital series designed to assist teachers in understanding the NCTM Process Standards and the ways in which they impact and guide student learning. An additional goal of this series is to provide practical ideas to support teachers as they ensure that the acquisition of process skills has a critical place in their math instruction. Through this series, teachers will gain an understanding of each process standard as well as gather ideas for bringing that standard to life within their math classrooms. It offers practical ideas for lesson development, implementation, and assessment that work with any curriculum. Each book in the series focuses on a critical process skill in a highlighted grade band and all books are designed to encourage reflection about teaching and learning. The series also highlights the interconnected nature of the process and content standards by showing correlations between them and showcasing activities that address multiple standards.

Students who develop an understanding of content skills and cultivate the process skills that allow them to apply that content understanding become effective mathematicians. Our goal as teachers is to support and guide students as they develop both their content knowledge and their process skills, so they are able to continue to expand and refine their understanding of mathematics. This series is a guide for math educators who aspire to teach students more than math content. It is a guide to assist teachers in understanding and teaching the critical processes through which students learn and make sense of mathematics.

Susan O’Connell
Series Editor
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The ways in which mathematical ideas are represented is fundamental to how people can understand and use those ideas.

—National Council of Teachers of Mathematics, Principles and Standards for School Mathematics

What Are Representations?

Doesn’t it sometimes seem as if students in primary math classes should all come equipped with hard hats and tool belts instead of just paper and pencils! At this early stage it is all about constructing important understandings about mathematical concepts and connecting those understandings, not just to one another but to the world around them. It isn’t about a single right answer to a missing addend problem; it isn’t merely about circling the right picture; it isn’t even about doing a whole page of addition or subtraction problems. Practice to improve fluency in computation is certainly important, as noted by NCTM’s Focal Points, but math at this level must include opportunities that are much richer. How is it that we can get at early student thinking, and what types of math representations can we model to promote students’ communication of that thinking? It is important to note that we can’t ask students to represent something that they haven’t yet learned. Are students at this stage even capable of somehow putting on paper the processes and thinking they have as they work through a number pattern or pattern of shapes? What does it truly mean to ask students to represent their mathematical thinking, and how can we support, model for, and encourage students to put on paper their interpretations of the mathematical models that helped them make those crucial connections between the concrete and the abstract?
Representing their solutions using pictures, diagrams, concrete models, numbers, and words is one of the means by which students can communicate to others their mathematical thinking and at the same time clarify in their own mind what meaning lies in the mathematics. Admittedly, at the primary level student-produced representations may be limited in scope and sophistication, but students should still be encouraged to create their own representations when appropriate. At the same time they need to be provided with multiple opportunities to see and experience representations that have already been created. As their ability to recognize multiple representations improves, so will their ability to produce multiple representations that communicate their own understandings. The goal is to increase students’ ability to construct meaning and at the same time improve their number sense. As teachers, we need to look for a variety of pictures, words, and concrete models that help our students construct meaning.

Manipulatives and concrete models provide students with important visuals that help to model the mathematics at hand, but they are just that—models. It is up to us as teachers to help students make the connections between the models and the mathematical concepts. Only by making those connections will students truly understand and internalize the math. Once those connections have been made, students can begin representing their own thinking in words, pictures, symbols, and diagrams and, later on, apply their knowledge to more complex problem-solving situations. Representations take many forms: an algorithm that can be used to represent a problem situation, a graph that represents data collected in the classroom, an array that shows a subtraction process, or diagrams that illustrate. Numbers, pictures, diagrams, equations, graphs, and models are all forms of mathematical representations. In the past, standard representations were simply taught to students, but teachers are now recognizing the power of alternative representations as tools through which students can explore and enhance their mathematical thinking.

The National Council of Teachers of Mathematics’ Principles and Standards for School Mathematics (2000) describes representations as fundamental to understanding and applying mathematics, and it makes three recommendations for using them in the mathematics classroom:

1. teachers should employ “representations to model and interpret physical, social, and mathematical phenomena” (NCTM 2000, 70);
2. students should be familiar with and comprehend various representations that can be used to describe phenomena; and
3. students should use mathematical representations to organize their thinking and reflect on numerical or geometric information.

Representations, instead of being taught to students, are valuable ways in which students can explore their own math thinking. Finding ways to represent their ideas pushes students to think more deeply about those ideas as they determine ways to communicate them to others. Representations give us insight into our students’ understandings by providing students with a way of working through their thought
processes. When students use base ten blocks to model addition or subtraction problems, draw circles or squares, or create a pictograph to organize and then analyze data, they are using various forms of representation to demonstrate their mathematical thinking. It is important that we give students many opportunities to represent their thinking and guide them to become proficient in representation. As they become more comfortable creating representations of their ideas, their mathematical thinking will greatly expand, as will their ability to communicate about that thinking.

Representation is both a process and a product by which students are able to explore and sort out mathematical concepts as well as communicate mathematically with their peers. Because representation is both a process and a product, students need many and varied opportunities to explore and sort mathematical concepts and to communicate these concepts to their peers. They need opportunities that will ultimately guide them toward more conventional forms of representation, facilitating their mathematical thinking and deepening their understanding. By enabling students to use a wide variety of representations, we are helping them to assemble a repertoire of tools from which to draw when exploring mathematical concepts.

What Is the Representation Standard?

Principles and Standards for School Mathematics (NCTM 2000) outlines standards for both math content and math processes. The content standards help us identify key math content that is critical to students’ understanding of mathematics, and the process standards help us identify those processes through which students learn and apply math content. Representation is a critical math process that supports students in their learning of math and their ability to express that learning.

In NCTM’s original standards document, Curriculum and Evaluation Standards for School Mathematics (1989), representation was included as a part of the communication standard, one of the four process standards in the original document. In the 2000 document, five process standards are outlined, with communication and representation individually addressed. NCTM now treats representation as its own process standard in order to address the broad scope of representation and its importance in learning mathematics. It has been recognized that the ability to represent ideas is fundamental to the study of mathematics.

In Principles and Standards for School Mathematics, NCTM recommends that instructional programs from prekindergarten through grade 12 should enable all students to—

■ create and use representations to organize, record, and communicate mathematical ideas;

■ select, apply, and translate among mathematical representations to solve problems; and

■ use representations to model and interpret physical, social, and mathematical phenomena.
The abilities to create representations to illustrate ideas, communicate thinking through representations, determine which representation would best fit a concept or idea, and use representation to model math situations are all critical components of this standard.

How Can Representation Support Student Learning?

Representation is both about helping students find their own ways to represent math ideas and about helping them understand conventional representations of math ideas (e.g., whole numbers, place value, fact families, patterns, charts, graphs, and diagrams). Students should be encouraged to represent their ideas and understandings in ways that make sense to them. Those representations provide key information for a teacher trying to determine where a student falls on the learning continuum.

Consider the following problem given to kindergarten, first and second grade students, each with a very different solution.

Use the numbers 14, 9, and 5 to draw a story problem.

No other conditions were set, and the students were encouraged to express their ideas using any operation or setting that made sense to them. By providing students with open-ended problems such as this, a teacher is able to gain more insight into the thinking of a student as his or her story unfolds. Asher, a kindergarten student, completed his story problem by first drawing an illustration of fourteen cars (Figure I–1). Since fourteen is the first number, that isn’t an unusual place for a kindergarten student to start. The interesting part of his work is that he skipped over the nine and used the five as the second part and crossed through five cars leaving nine. Once he had his problem drawn, Asher wrote a descriptive sentence that matched his drawing completely. Even though Asher didn’t show a connection to the algorithm, his drawing and his sentence show an understanding, even at this early age, of the process of subtraction. No doubt, with questioning, Asher would be able to supply the rest.

Ryan, a first-grade student, chose to use the three numbers to create an addition problem, combining the two smaller numbers to equal the third. His realistic representation of the bugs shows his ability to connect the math to a real-world situation, and he supported his picture with the appropriate numbers (Figure I–2). His written description of the situation provided information on what he was thinking as he completed the task. At this point Ryan has demonstrated that he has a sense of what it means to join two groups together to make one. This process of joining numbers is one of the very basic structures in early addition. With some further questioning, Ryan was able to orally communicate additional solutions using the three numbers.

Devin, a second-grade student, had a somewhat different approach to the same problem. Instead of using a realistic drawing, he chose to use an iconic representation (circles) for his solution. He extended the numbers to include all of the combinations possible for this fact family by showing all of the possible combinations of addition
and subtraction problems (Figure 1–3). While his pictorial representation and written communication support only one of the problems, he obviously has a sense of the relationship these numbers have to one another, and he was able to communicate that relationship numerically.

Are these different representations examples that you might expect from primary students? Does the thinking represented in any of them provide any clues to the level of understanding and conceptualization they have of the number facts or the numbers themselves, and has the representation itself helped to support their learning? In this case, Devin certainly seems to have an understanding of how the numbers are related, and he demonstrated that understanding by including the four possible combinations. Ryan also seems to have a clear picture of the relationship of the three numbers, and the fact that he chose to start with nine instead of fourteen is significant. Most students will start their problems with the first number they are given. Ryan, however, chose to
start with the second number. Allowing time for oral communication in a problem such as this, when specific written communication may not be possible, is an important step in getting at a student’s thinking. Although many representations are conventional, not all students will show their thinking in conventional ways. It is important that we allow them to work through their thinking in ways that make sense to them and then guide them toward more conventional representations to assist them in better understanding mathematics, as well as to enable others to understand their thought processes.

In this book you will see many examples of ways in which you can provide students with opportunities to represent their thinking and of how to enable them to use those representations in meaningful ways. The process of representation, which is the act of putting one’s ideas into words, pictures, numbers, or symbols, is as important as the product: the actual representation of those ideas. We must provide students with many opportunities and ongoing support as they attempt to represent their own ideas and as they explore standard ways to represent math ideas.
In addition to providing students with many opportunities to represent their thinking, we must expose them to a variety of representations. When teaching the concept of subtraction with regrouping, Mr. Wilson chose to provide his students with base ten blocks that they could manipulate to illustrate the need for regrouping. At the same time, he modeled the processes of regrouping using interactive white board technology. As he gives his students a problem to work through, they can see if their manipulatives mirror the ones Mr. Wilson is using. It is important, too, to include in our instruction other ways of representing subtraction, such as pictures, numbers, and words. In addition, as students' mathematical thinking matures, they learn to choose more alternate ways of representing their ideas and information. For example, when faced with a set of data, a second-grade student might need to decide how to graphically represent the information on a pictograph. A first-grade student being asked to demonstrate understanding of the difference between odd and even numbers may need to use tiles or ten frames to represent the differences. By encouraging children to

Figure 1–3  Devin’s targeted problem using iconic representation.
use multiple representations, we help them build a repertoire from which to choose strategies for solving mathematical problems.

Finally, students should use representations to model real-world phenomena. One student might record the daily temperature in her town each day for a week and then represent the data in a table, whereas a student interested in learning which lunch choice is most popular in the school cafeteria might collect data from classmates and then use a real graph to represent these findings to peers, teachers, and the principal. Providing students with opportunities to model different phenomena in the world around them leads to a better understanding of that world and to the mathematical relationships within it.

**Does Your Classroom Look Like a Math Classroom?**

Does your classroom speak math, and would someone be able to recognize it as a classroom that places an emphasis on math talk? Many of us have worked hard to have colorful wall displays and bulletin boards made from commercially produced cutouts, but what may be missing is a classroom that clearly says, “This is a math classroom, and math is spoken here.” Having a room that clearly demonstrates the importance of sharing students’ mathematical ideas and work goes a long way toward creating a culture that encourages student discourse in the area of mathematics. As teachers, we want to be certain students realize that their math writing and drawings are the most important part of what goes on in that classroom on a daily basis.

Building important vocabulary is critical at this point. As teachers, we can go from merely asking students to define math concepts using the glossary in the back of their text to having them write the definitions in their own words, representing each concept using an illustration of their choice, and finally adding a real-life application that is appropriate for the concept. Moving the class from copying dictionary definitions or repeating a given definition to representing concepts in multiple forms is not an easy process. Modeling the process in the beginning is essential, and as students begin to feel more comfortable with this new way of representing the math, they will look forward to creating their own representations, which for them brings real meaning to mathematics. Each week students can help generate a list of two to three terms discussed during the week or terms that may have come up during math lessons. Students can then be encouraged to discuss these words in small groups and plan how they think the concepts could best be represented. Yes, students at this level can learn from one another in a small group setting. By working cooperatively on their representations, students will begin to engage more and more in math conversations, which will help other students clarify concepts that may still be somewhat fuzzy.

At the same time, students may begin to look for more and more ways to demonstrate the real-life applications of the math for their own representations. A wonderful added benefit to this process might become evident at parent conferences. Parents may begin sharing with you ways in which their child is sharing the vocabulary at home. Imagine parents being asked by their child to look at work for examples of math concepts or examples in a newspaper or on the Internet. This home–school connection can be an unexpected bonus of the process and a very welcome communication tool.
that helps parents keep up to date on topics covered during the week. You may also find that by challenging the students to find the real-life applications, you now have instant wall displays that became sources of conversation around mathematics. There is no doubt about what is valued in your classroom.

Creating a Classroom Environment That Encourages Multiple Student Representations

Teachers can facilitate the process of student representations by creating a classroom that encourages students to represent their math thinking in a variety of ways. One way to encourage a climate of student representation is to provide students with the tools they need to create those representations. Baskets of markers, crayons, and colored pencils to which students have easy access will go a long way toward getting students motivated to express their ideas. Because most students in the beginning see representations as drawings, they may be more inclined to create those drawings if they have drawing tools. It would be great if every classroom had unlimited supplies of easel pads so that students could share their representations for the whole class to see, but we all know that isn’t the case. One possible solution is to have bulletin boards covered with brown wrapping paper, or any other solid-colored paper, and allow students to draw their representations directly on the board.

It is also important that students have an opportunity to share their interpretations of the mathematics with those around them and discuss how their representations help them to understand and illustrate the concepts. A room that encourages oral communication and cooperative work either in small groups or in pairs is ideal. When students are able not only to discuss the mathematics with their peers but to listen to them explain their thinking, they are more likely to bring their own meaning to the concepts at hand.

CLASSROOM-TESTED TIP

As students are working on representing their math ideas, consider putting four desks together and placing a precut piece of bathroom tile board (from your local building supply store) on top of the four desks, creating an instant work space. The tile board (made of melamine) should have the same dimensions as the four desks combined. Any manipulatives students may be using will not fall through the cracks, and more important, with a dry-erase marker, students can create their representations right on the table before they put their finished copy on paper. You can do this for pairs of student desks as well. Students working on the tile board can all do the problem in their own way and then share at their table. This also makes a great “gallery walk”: students can get up and walk around the room, looking at all of the other solutions and representations of the problem.

A note of warning: smooth the edges of the tile board (using a router or sandpaper) prior to use to eliminate sharp edges and corners, and if the boards are heavy, help students put them on top of their desks.
How Can the Use of Representations Help All Populations?

Classrooms today are becoming increasingly diverse. With added ELL (English language learner) populations as well as the increasingly inclusive nature of all classrooms, it is more important than ever to meet the needs of a wide range of student abilities. At the same time, teachers are becoming more cognizant of the research associated with learning styles, and finding ways to reach those learners is crucial. Teachers who encourage students to represent their mathematical thinking in a variety of ways can help meet the needs of those students who may have difficulty explaining or writing down their thinking in words. Students needing alternate ways of expressing their thinking oftentimes find that pictorial representations open a much-needed path for communication. For example, an ELL student will find it much easier to solve a problem that is presented to him pictorially than written out. Likewise, a teacher is much more likely to see a student’s thinking when the steps in her solution are either drawn out or written in numeric fashion. The same holds true for special education students working in an inclusive classroom. When the entire class is involved in looking at the math through mathematical models and being encouraged to represent their answers in like fashion, all students feel included and do not feel that they are being singled out for alternate instruction. The key is for all students to feel comfortable with this process and to feel equal in their ability to understand and communicate about mathematics. When asked to “explain their answer” and “show their work,” students need to understand that there are a number of ways in which they can do so.

CLASSROOM-TESTED TIP

As students become more comfortable sharing their ideas with others, it is important that all students have an opportunity to participate in an equal fashion. For that reason, equity sticks are a useful tool in any classroom. From a craft department, purchase craft sticks at the beginning of the year and write each student’s name on one of the sticks. As you work through a lesson and question the students, reach into the container, pull out an equity stick, and call on the student whose name is written on the stick.

A word of caution in using the sticks: If you have a reluctant learner or a shy student who may not want to answer out loud for fear of being wrong, work out a signal ahead of time so the student can communicate whether or not he or she wants to answer prior to you calling on this student. If you see this signal and know the student feels comfortable enough to answer out loud, then no matter whose name is on the stick you pulled out of the cup, you can call on your reluctant learner. As an added bonus, equity sticks are a great way to help teachers learn students’ names at the beginning of the year.
Considerations in Lesson Planning

Ongoing attention to lesson design also helps to create an environment in which students feel comfortable representing their math ideas with pictures, tables, numbers, or manipulatives. Planning lessons that routinely include higher-order questioning, cooperative group work, and class discussions set the stage for productive representation. Using manipulatives, posing problems, and reading math-related literature each provide a stimulus for representing math ideas. A balance of whole-group activities, small teacher-led groups, cooperative groups, partner work, and independent tasks provides varied opportunities for students to develop their representation skills.

In planning lessons, it is important to allow enough time for students to explore math ideas and ways to represent those ideas. Asking follow-up questions that uncover students’ reasoning or procedures is vital. Posing fewer tasks but allocating time to discuss ideas and solutions builds students’ reasoning and problem-solving skills. Structuring assignments with fewer rote tasks and more reasoning helps balance students’ skill development. To ensure that students are not hurried, allow enough time for them to share ideas with others prior to beginning an independent task or spend a few minutes modeling representations prior to assigning an independent task.

Creating an environment that promotes communication is about modifying our expectations from quiet students to verbal students and from correct answers to reflective thinking. It is about developing a community of learners who respect each other’s ideas, whether right or wrong, and who work to support each other in building math understanding.

How This Book Will Help You

This book is designed to help you better understand the representation process standard and its significance. Although the book is specifically designed for teachers in prekindergarten through grade 2 and correlates with math content generally taught at those levels, teachers at other grade levels may find strategies and activity ideas that can be used with their students as well. In each section the standards are explained and illustrated through a variety of student work samples, practical ideas are shared for helping students develop their skills in representing mathematical information, and tools to assist you in assessing students’ representations are presented.

In Chapter 1, “How Representations Support Learning,” a variety of strategies are presented that illustrate the importance of providing students with opportunities to create representations that make math meaningful to them. The chapter also offers some suggested interpretations teachers might make about the levels of student understanding just from looking at student representations. In addition, you will find some practical applications for getting at student representations for various strands of mathematics such as the use of pictorial representations in a variety of problem-solving situations and at various levels of cognitive demand (Figure I–4). Logic problems and their significance in helping students organize solutions in a systematic way are discussed.
Chapter 2, “Using Manipulatives to Model and Illustrate Key Math Concepts,” takes a closer look at how the use of manipulatives can guide students as they develop their own representations for the math at hand. Manipulatives are a staple in all math classes. Whether these tools are store-bought or teacher created, the research is clear on their importance. Whether it is in geometry or number computation, manipulatives can provide students with a model that will help them internalize a concept and then enable them to create their own model and interpretation of the math concept. This chapter also provides classroom examples of how the connection between the concrete manipulative and the student representation can be used to build student understanding.

Chapter 3, “Using Pictures and Diagrams to Represent Mathematical Thinking,” takes you through some of the possible ways to get students to make use of a hundreds chart to develop a sense of number relationships. Transitioning from manipulatives to pictorial representations is a key component of this chapter, and examples are provided for ways of making use of student drawings to assess conceptual un-

Figure I–4  How many snowmen tall are you?
nderstandings. In addition, using a key reading strategy such as “before,” “during,” and “after,” teachers may find a way to provide their students with an avenue for representing mathematics that will help them build on prior knowledge and connect that knowledge to new processes.

In Chapter 4, “Using Numbers and Symbols to Represent Mathematical Ideas,” the value of student-invented algorithms is explored as well as ways to use them to support student learning. Moving students from pictorial to numeric representations of the mathematics is ultimately the goal in any math classroom, but helping students make the connections and bring meaning to the math is a crucial process that cannot be rushed.

Developing student understanding of how to graph specific types of data is discussed in Chapter 5, “Using Tables and Graphs to Record, Organize, and Communicate Ideas.” Whether graphing information on a picture graph or a bar graph, students need to understand and see the various data representations used so that they have the skill to choose the appropriate representations for any type of data. This chapter also shows ways to help students begin organizing solutions using tables.

Chapter 6, “Assessing Students’ Representations,” looks at how teachers can make decisions about assessments as they apply to the representation standard and suggestions for ways in which student representations can be assessed.

Finally, in Chapter 7, “Representation Across the Content Standards,” we share lesson ideas to illustrate the representation process standard as it connects to the teaching of numbers and operations, algebra, geometry, measurement, and data and probability. Engaging students in representing their mathematical thinking in all content standards will ensure their success in applying the process in other areas.

At the end of each chapter we include reflection questions that can be used for individual reflection or to generate faculty study group discussions. The accompanying CD provides a variety of practical resources you can use to help your students more effectively represent their math ideas, including lesson ideas that can easily be adapted to meet your individual needs. Some of the activities are in the format of a worksheet and others appear as a set of teacher directions that provide lesson seeds for use in the classroom. Some activities are more teacher-directed than others, and in some cases you will need to provide additional directions to aid students as they work through the problems. This book will expand your understanding of the representation standard and will provide you with the practical resources that you’ll need to implement the ideas with your students. You will also be able to personalize the activities on the CD for use in your classroom. All along the way, the “Classroom-Tested Tip” boxes provide examples of time-saving, student-motivating, and curriculum-enhancing ideas that come from the classrooms of experienced math teachers. The tips are designed to make classroom management and instruction a little easier and more meaningful and to send the message that, yes, math can be fun!

Each chapter builds on the previous chapter but can also stand alone if you are looking for one or two ideas to help math instruction in your classroom. As you read, keep this in mind: This book is written by math teachers for math teachers, with the hope that as you read at some point in time you’ll have that “aha” moment and will say to yourself, “That just might work!”
Questions for Discussion

1. What does it mean to say that representation can be both a process and a product?

2. How can representing math ideas help students strengthen their understanding of math?

3. How can attention to students’ representations help teachers better assess students’ understanding?

4. What are some of the physical characteristics of a math classroom where students are encouraged to freely represent math ideas?

5. How can students be encouraged to communicate mathematically through representations?

6. How can the classroom environment encourage a climate of student representation?
How Long Is It?

The Task

Students will explore linear measurement using nonstandard units and create a bar graph of their results.

Materials

• Containers of Unifix cubes, one per pair of students
• Recording sheet, one per pair of students
• Index cards with pictures of classroom items on them
• Chart paper to create pictograph

Directions

1. Introduce students to the concept of measuring length after reading an appropriate piece of children's literature, such as Inchworm and a Half by Elinor Pinczes. Ask students what could be used to measure how long something is. Examples include a ruler, a yardstick, and a measuring tape.
2. Tell students that they will be measuring objects in the classroom with Unifix cubes to determine how long the objects are. Model how to measure the length of one side of a table. Have a student place Unifix cubes along the length of the table and then count the cubes to determine the length of one side of the table. Model again using the width of the door.
3. Pass out the index cards with pictures of classroom items, and give a container of Unifix cubes to each set of partners. Have students measure the length of each object shown on the cards and record their findings on a recording sheet. After all students have measured the objects, record the results on chart paper.
4. Create a pictograph using Unifix cubes to compare the lengths of the different items measured.
Discussion

Try these questions to stimulate thinking and discussion:

- Which object was the longest? How do you know?
- Which object was the shortest? How do you know?
- How does ___ compare with ___? How do you know?
- Which objects are longer than _____? How do you know?
- Which objects are shorter than _____? How do you know?

Tiered Learning

Adjust or extend the activity in the following ways:

- Simplify the task by giving students a Unifix cube train of a certain length (for example, 10 cubes), and have students find something in the room that is that length.
- Challenge students to order the objects from shortest to longest. Have them compare the longest item and the shortest item: How much longer is ___ than ___?
- Challenge students to create a bar graph from the class data.
Name ____________________________________________

How Long Is It?

We measured ________________________________.

It is ______________________ Unifix cubes long.